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Nonlinear evolution of small disturbances into roll waves in an inclined open channel

By J. YU[1] AND J. KEVORKIAN[2]

[1] Department of Mathematics and Statistics, University of Vermont, Burlington, VT 05405, USA

[2] Department of Applied Mathematics, University of Washington, Seattle, WA 98195, USA

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This paper concerns the asymptotic behaviour (for the limiting case of small amplitudes) of small disturbances as they evolve in time to produce the quasi-steady pattern of roll waves first discussed by Dressler in 1949. Roll waves exist if F, the undisturbed Froude number (dimensionless speed) of the flow, exceeds 2, and consist of a periodic pattern of bores separating two special continuous solutions of the governing equations in a uniformly translating frame. The mathematical problem is rather interesting as solutions of the linearized equations are unstable for $F > 2$. Thus, it is crucial to account for the cumulative effect of small nonlinearities to obtain a correct description of the flow over long times. We concentrate on the weakly unstable problem ($0 < F-2 \ll 1$) and use multiple scale expansions to derive the dominant evolution equation that governs the solution behaviour for long times. This turns out to be an integro–partial differential equation of first order that we solve numerically in conjunction with the jump condition that follows from the exact bore conditions. We present asymptotic and numerical results for periodic as well as isolated initial disturbances, and show that our theory predicts the solution accurately for both the transient and quasi-steady phases.

1. Introduction

We consider nonlinear waves on a thin fluid layer flowing down an inclined open channel. As the Navier–Stokes equations with the nonlinear free-surface boundary conditions are intractable even using numerical simulation, approximate solutions for the limiting cases of low and high Reynolds number are quite useful. At low Reynolds number, the local velocity profile can be approximated to leading order by the viscous Nusselt profile, leading to a single nonlinear partial differential equation for the normal interfacial height. This type of problem is studied by Nakaya (1975), Kawahara & Toh (1985) and Chang (1986).

At high Reynolds number, a macroscopic averaging technique assuming a flat velocity profile and a hydrostatic pressure balance has been used by Dressler (1949), Whitham (1974) and Needham & Merkin (1984). Using dimensionless variables, the equations for mass and momentum conservation for shallow water in a broad, slightly inclined channel are given by (Kevorkian 1990, §5.1.1)

$$h_t + uh_x + hu_x = 0, \tag{1.1a}$$

$$h(u_t + h_x + uu_x) = h - u^2/F^2. \tag{1.1b}$$

As shown in figure 1, u is the flow speed parallel to the channel bottom averaged over

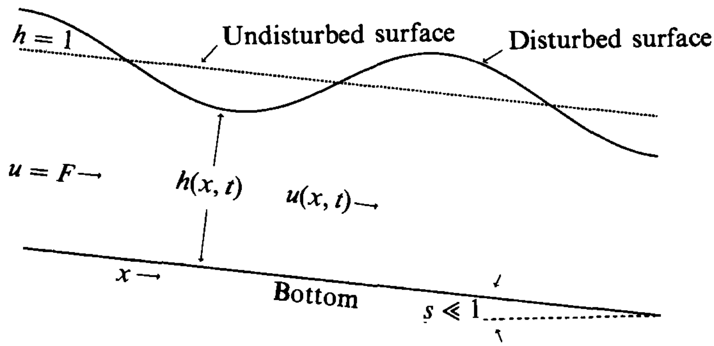

FIGURE 1. Flow geometry.

h, and h is the height of the free-surface normal to the bottom. The first and second terms on the right-hand side of (1.1b) respectively represent the gravitational and friction forces on a column of water of width dx. The Froude number F is the dimensionless speed of undisturbed flow of unit height with viscous and gravitational forces in perfect balance. The choice of the horizontal and vertical lengthscales and timescale leading to the above dimensionless formulation implies that $F = ((\tan s)/C)^{\frac{1}{2}}$, where $\tan s$ is the channel slope and C is the friction coefficient. Thus F increases with increasing slope or decreasing friction, as expected.

As was pointed out by Whitham (1974), the uniform flow is locally unstable when F is larger than 2. That is, for any small disturbance imposed on the uniform flow, the response predicted by linear theory for (1.1) has an amplitude that grows exponentially in time. However, solutions of the nonlinear equations (1.1) are stable. In fact, various levels of approximation of the flow equations may be used to exhibit bounded quasi-steady solutions; these are time-independent periodic solutions in a coordinate frame moving downstream with a constant speed. In particular, the dimensional form of (1.1) was used in Dressler's (1949) study. He showed that no continuous quasi-steady solutions exist, and that it is necessary to have $F > 2$ in order to find discontinuous quasi-steady periodic solutions that he called *roll waves*. In this case each cycle in the periodic pattern of roll waves consists of two special continuous solutions joined by a bore. A formulation, including an additional hu_{xx} term in (1.1b) to account for energy dissipation, was analysed by Needham & Merkin (1984) to show that in this higher-order model periodic quasi-steady continuous waves exist when the uniform flow is locally unstable. These results were independently confirmed by Hwang & Chang (1987) using normal form techniques.

In this paper we present an analysis for the weakly nonlinear wave evolution using a multiple scale asymptotic expansion where the small parameter ϵ measures the amplitude of the initial disturbance. In particular, we consider the initial conditions

$$h(x,0;\epsilon) = 1+\epsilon h_0(x), \tag{1.2a}$$

$$u(x,0;\epsilon) = F+\epsilon u_0(x), \tag{1.2b}$$

for prescribed disturbance functions $h_0(x)$ and $u_0(x)$, and we discuss two specific examples corresponding to periodic or isolated disturbances. The time-dependent solution that we derive takes into account the cumulative effect of the small nonlinearities, and tends to Dressler's roll-wave solution as $t \rightarrow \infty$ for the case of periodic disturbances. For an isolated initial disturbance, we find a limiting solution (as $t \to \infty$) that is a special case of Dressler's quasi-steady solution in the form of a single roll wave. We also compare our results with numerical integrations of the exact problem (1.1)–(1.2) in each case, and show that the transient behaviour of the solution including the intensity and propagation of discontinuities is accurately described. We concentrate on weakly unstable flows by restricting the Froude number to the one-parameter family $F = 2+\alpha\epsilon$, where α is a positive $O(1)$ constant. The strongly unstable problem, where $F-2 = O(1)$, is not discussed because a multiple scale analysis gives a solution valid for times that are only marginally longer than $O(1)$. The effect of the strong instability of initial disturbances is to produce a steady state that, although bounded, differs by $O(1)$ from the underlying uniform flow; thus, no perturbation scheme can be expected to describe such a solution over long times.

The general multiple scale procedure that we use here is analogous to that discussed in Kevorkian & Yu (1989); one new ingredient is that in the present case the linearized solution is unstable as $t \to \infty$. The example that we study is a special case of the mathematical problem defined by a system of two quasi-linear hyperbolic equations with linearly unstable solutions in a given parameter range. The essential new result that emerges from our analysis is that the evolution of the growing disturbance now obeys an integro–partial differential equation that describes the steepening of the wave and eventual bore formation. As in earlier work, the basic conservation laws of the governing system imply appropriate jump conditions for such discontinuities in the evolution equations.

2. The linearized solution; stability

We begin our discussion by studying the behaviour of the linear problem governing perturbations to the uniform flow solution ($h = 1$, $u = F$) of (1.1). The following pair of quasi-linear equations for the dependent variables v_1, v_2:

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{2} A_{ij}(v_1, v_2)\frac{\partial v_j}{\partial x} = f_i(v_1, v_2); \quad i = 1, 2 \tag{2.1}$$

for a given matrix $\{A_{ij}(v_1, v_2)\}$ and column vector $f_i(v_1, v_2)$, is a general version of the system (1.1). Let the constant state $v_1 = v_1^{(0)} = \text{const.}; v_2 = v_2^{(0)} = \text{const.}$ be a solution of (2.1), i.e. $f_i = (v_1^{(0)}, v_2^{(0)}) = 0$ for $i = 1, 2$, and look for a solution that is close to this constant state. This would be the case, for example, if one wishes to solve the initial-value problem $v_i(x, 0) = v_i^{(0)} + \epsilon b_i(x)$; $i = 1, 2$ with $0 < \epsilon \ll 1$ and prescribed functions $b_i(x)$. This perturbation problem is a special case of the situation discussed in Kevorkian (1990, §4.5). It is easily seen that the linearized equations for the perturbation terms $u_i(x, t)$:

$$v_i(x, t; \epsilon) = v_i^{(0)} + \epsilon u_i(x, t) + O(\epsilon^2) \tag{2.2}$$

are

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{2} A_{ij}^{(0)} \frac{\partial u_j}{\partial x} + \sum_{j=1}^{2} B_{ij} u_j = 0; \quad i = 1, 2, \tag{2.3a}$$

where $\{A_{ij}^{(0)}\}$ and $\{B_{ij}\}$ are the constant matrices

$$A_{ij}^{(0)} = A_{ij}(v_1^{(0)}, v_2^{(0)}); \quad B_{ij} = -\frac{\partial f_i}{\partial v_j}(v_1^{(0)}, v_2^{(0)}). \tag{2.3b, c}$$

For the hyperbolic problem we have $(A_{11}^{(0)}-A_{22}^{(0)})^2+4A_{12}^{(0)}A_{21}^{(0)}>0$, and it follows that the eigenvalues λ_1, λ_2 of $\{A_{ij}^{(0)}\}$ are real and distinct:

$$\lambda_{1,2}=\tfrac{1}{2}\{A_{11}^{(0)}+A_{22}^{(0)}\pm[(A_{11}^{(0)}-A_{22}^{(0)})^2+4A_{12}^{(0)}A_{21}^{(0)}]^{\frac{1}{2}}\}, \tag{2.4}$$

and we may transform (2.3*a*) to the characteristic form

$$\frac{\partial U_i}{\partial t}+\lambda_i\frac{\partial U_i}{\partial x}+\sum_{j=1}^{2}C_{ij}U_j=0; \quad i=1,2. \tag{2.5}$$

Here the U_i are defined by the linear transformation

$$u_i=\sum_{j=1}^{2}W_{ij}U_j; \quad i=1,2 \tag{2.6}$$

in terms of the matrix

$$\{W_{ij}\}=\begin{pmatrix}-A_{12}^{(0)} & -A_{12}^{(0)}\\ A_{11}^{(0)}-\lambda_1 & A_{11}^{(0)}-\lambda_2\end{pmatrix} \tag{2.7}$$

and $\{C_{ij}\}$ is the transformed matrix $\{B_{ij}\}$:

$$\{\boldsymbol{C}\}=\{\boldsymbol{W}^{-1}\}\{\boldsymbol{B}\}\{\boldsymbol{W}\}. \tag{2.8}$$

A solution of the linearized problem (2.3) or (2.5) is said to be stable if an initial disturbance remains bounded in the far field (i.e. if $t\to\infty$ along either characteristic). As shown in Kevorkian (1990), the necessary conditions for stability are $C_{11}>0$ and $C_{22}>0$ if $C_{11}\neq 0$, $C_{22}\neq 0$; and $C_{12}C_{21}>0$ if $C_{11}=C_{22}=0$.

For the case of the system (1.1), we have $v_1=h$; $v_2=u$; $v_1^{(0)}=1$; $v_2^{(0)}=F$, and denoting u_1 and u_2 in (2.3) by $\tilde{h}(x,t)$ and $\tilde{u}(x,t)$, this linearized system becomes

$$\tilde{h}_t+F\tilde{h}_x+\tilde{u}_x=0, \tag{2.9a}$$

$$\tilde{u}_t+\tilde{h}_x+F\tilde{u}_x-\tilde{h}+2\tilde{u}/F=0. \tag{2.9b}$$

The eigenvalues are $\lambda_1=F+1$; $\lambda_2=F-1$, and the characteristic form (2.5) has the matrix components $C_{11}=C_{21}=1/F-\frac{1}{2}$; $C_{12}=C_{22}=1/F+\frac{1}{2}$. For $F>0$, the stability condition $C_{22}>0$ always holds, and the condition $C_{11}>0$ requires $F<2$.

Although one can solve (2.9) for arbitrary initial conditions in integral form in terms of the Bessel function J_0, the result is rather complicated; the qualitative behaviour of the solution as $t\to\infty$ can only be discerned after further asymptotic analysis. However, it is possible to derive an explicit result for periodic initial conditions, and we begin with the choice

$$\tilde{h}(x,0)=a_1\cos x+a_2\sin x, \tag{2.10a}$$

$$\tilde{u}(x,0)=a_3\cos x+a_4\sin x. \tag{2.10b}$$

We assume a solution in the form

$$\tilde{h}(x,t)=\alpha_1(t)\cos\sigma+\alpha_2(t)\sin\sigma, \tag{2.11a}$$

$$\tilde{u}(x,t)=\alpha_3(t)\cos\sigma+\alpha_4(t)\sin\sigma, \tag{2.11b}$$

where $\sigma=x-Ft$, and upon substituting (2.11) into (2.9) we obtain a fourth-order system for the $\alpha_i(t)$ which can be solved, subject to $\alpha_i(0)=a_i$, in the form

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}=\mathbf{e}^{Mt}\begin{pmatrix}a_1\\ a_2\\ a_3\\ a_4\end{pmatrix}, \tag{2.12}$$

where the matrix $\mathbf{e}^{Mt}$ is defined by

$$\mathbf{e}^{Mt} = \frac{1}{2(\mu^2+\beta^2)}[\mathrm{e}^{\gamma^+ t}(\sin\beta t\boldsymbol{S}_1+\cos\beta t\boldsymbol{R}_1)+\mathrm{e}^{\gamma^- t}(\sin\beta t\boldsymbol{S}_2+\cos\beta t\boldsymbol{R}_2)], \quad (2.13)$$

and we have used the notation

$$\mu = \left[\frac{1}{2}\left\{\left[1+\left(1-\frac{1}{F^2}\right)^2\right]^{\frac{1}{2}}-1+\frac{1}{F^2}\right\}\right]^{\frac{1}{2}}, \quad \beta = \left[\frac{1}{2}\left\{\left[1+\left(1-\frac{1}{F^2}\right)^2\right]^{\frac{1}{2}}+1-\frac{1}{F^2}\right\}\right]^{\frac{1}{2}}, \quad (2.14a, b)$$

$$\gamma^+ = -\frac{1}{F}+\mu, \quad \gamma^- = -\frac{1}{F}-\mu, \quad (2.14c, d)$$

$$\boldsymbol{S}_1 = \begin{pmatrix} \beta/F & -\kappa-\mu/F & -\mu & -\beta \\ \kappa+\mu/F & \beta/F & \beta & -\mu \\ -\mu+\beta & -\mu-\beta & -\beta/F & -\kappa+\mu/F \\ \mu+\beta & -\mu+\beta & \kappa-\mu/F & -\beta/F \end{pmatrix}, \quad (2.14e)$$

$$\boldsymbol{R}_1 = \begin{pmatrix} \kappa+\mu/F & \beta/F & \beta & -\mu \\ -\beta/F & \kappa+\mu/F & \mu & \beta \\ \mu+\beta & -\mu+\beta & \kappa-\mu/F & -\beta/F \\ \mu-\beta & \mu+\beta & \beta/F & \kappa-\mu/F \end{pmatrix}, \quad (2.14f)$$

$$\boldsymbol{S}_2 = \begin{pmatrix} \beta/F & \kappa-\mu/F & -\mu & -\beta \\ -\kappa+\mu/F & \beta/F & \beta & -\mu \\ -\mu+\beta & -\mu-\beta & -\beta/F & \kappa+\mu/F \\ \mu+\beta & -\mu+\beta & -\kappa-\mu/F & -\beta/F \end{pmatrix}, \quad (2.14g)$$

$$\boldsymbol{R}_2 = \begin{pmatrix} \kappa-\mu/F & -\beta/F & -\beta & \mu \\ \beta/F & \kappa-\mu/F & -\mu & -\beta \\ -\mu-\beta & \mu-\beta & \kappa+\mu/F & \beta/F \\ -\mu+\beta & -\mu-\beta & -\beta/F & \kappa+\mu/F \end{pmatrix}, \quad (2.14h)$$

and

$$\kappa = \mu^2+\beta^2. \quad (2.14i)$$

We note that γ^- is negative for all F and γ^+ is positive or negative depending on whether $F > 2$ or $F < 2$ respectively. Thus, as expected, the initial disturbances decay exponentially if $0 \leqslant F < 2$ whereas if $F > 2$ there is one exponentially growing disturbance.

To fix ideas, let us simplify the foregoing result by choosing $a_1 = a_3 = a_4 = 0$ and $a_2 = -1$. We then find the following expressions for $\tilde{h}(x,t)$ and $\tilde{u}(x,t)$:

$$\tilde{h}(x,t) = -\frac{1}{2(\mu^2+\beta^2)}\left\{\mathrm{e}^{\gamma^+ t}\left[\left(\mu^2+\beta^2+\frac{\mu}{F}\right)\sin\zeta^+ +\frac{\beta}{F}\cos\zeta^+\right] + \mathrm{e}^{\gamma^- t}\left[\left(\mu^2+\beta^2-\frac{\mu}{F}\right)\sin\zeta^- -\frac{\beta}{F}\cos\zeta^-\right]\right\}, \quad (2.15a)$$

$$\tilde{u}(x,t) = -\frac{1}{2(\mu^2+\beta^2)}\{\mathrm{e}^{\gamma^+ t}[(\beta+\mu)\sin\zeta^+ +(\beta-\mu)\cos\zeta^+] + \mathrm{e}^{\gamma^- t}[-(\beta+\mu)\sin\zeta^- -(\beta-\mu)\cos\zeta^-]\}, \quad (2.15b)$$

where $$\zeta^+ = x-(F+\beta)t = \sigma-\beta t, \quad \zeta^- = x-(F-\beta)t = \sigma+\beta t. \tag{2.16a, b}$$

Thus, the solution consists of two sinusoidal waves with time-dependent amplitudes travelling with speeds $C^+ = F+\beta$ and $C^- = F-\beta$. If $0 \leqslant F < 2$, both waves decay exponentially as $t \to \infty$. If, however, $F > 2$, the wave with speed C^+ has an amplitude which grows exponentially like $e^{\gamma^+ t}$, while the C^- wave still decays.

Because of linearity, the above procedure generalizes to arbitrary periodic initial data after superposition of the results for each mode. The exponential growth predicted by the linear theory for $F > 2$ is not indicative of the actual solution as we have totally ignored the effect of small nonlinear terms. These terms eventually become important whenever the linear theory predicts growth. In fact, the existence of a bounded quasi-steady solution of the nonlinear problem is well known, Dressler (1949), and we consider next how initially unstable disturbances evolve to form a quasi-steady roll wave pattern as $t \to \infty$.

3. Weakly unstable solutions $(F-2) \ll 1$

We restrict our attention to the weakly unstable problem $0 < F-2 \ll 1$ as in this case we can explicitly derive all the qualitative features of the more general case $F > 2$, including the steepening of waves, bore formation, and evolution of a quasi-steady pattern.

We assume $F = 2+\epsilon\alpha$, where α is a positive $O(1)$ constant and look for a multiple scale expansion of (1.1) in the form

$$h(x,t;\epsilon) = 1+\epsilon h_1(x,t,\tilde{t})+\epsilon^2 h_2(x,t,\tilde{t})+O(\epsilon^3), \tag{3.1a}$$

$$u(x,t;\epsilon) = 2+\epsilon[\alpha+u_1(x,t,\tilde{t})]+\epsilon^2 u_2(x,t,\tilde{t})+O(\epsilon^3), \tag{3.1b}$$

where $\tilde{t} = \epsilon t$ is the slow time, and the form of (3.1b) implies that ϵu_1 is the perturbation to $2+\epsilon\alpha$.

Substituting (3.1) into (1.1a, b) shows that h_1 and u_1 must satisfy

$$h_{1_t}+2h_{1_x}+u_{1_x} = 0, \tag{3.2a}$$

$$u_{1_t}+h_{1_x}+2u_{1_x}-h_1+u_1 = 0, \tag{3.2b}$$

and h_2, u_2 must satisfy

$$h_{2_t}+2h_{2_x}+u_{2_x} = -h_{1_{\tilde{t}}}-(\alpha+u_1)h_{1_x}-h_1 u_{1_x}, \tag{3.3a}$$

$$u_{2_t}+h_{2_x}+2u_{2_x}-h_2+u_2 = -u_{1_{\tilde{t}}}-(\alpha+u_1)u_{1_x}+\tfrac{1}{2}\alpha u_1-(h_1-\tfrac{1}{2}u_1)^2. \tag{3.3b}$$

It is very helpful in the analysis to introduce the characteristic dependent and independent variables of the linearized problem:

$$R_1 = h_1-u_1; \quad S_1 = h_1+u_1, \tag{3.4a, b}$$

$$\xi = x-3t; \qquad \eta = x-t. \tag{3.4d, e}$$

Equations (3.2) then become

$$R_{1_\xi}-\tfrac{1}{2}R_1 = 0; \quad S_{1_\eta}-\tfrac{1}{2}R_1 = 0, \tag{3.5a, b}$$

and can be solved sequentially. Integrating (3.5a) first we have

$$R_1(\xi,\eta,\tilde{t}) = f_1(\eta,\tilde{t})\,e^{\xi/2}. \tag{3.6a}$$

Substituting (3.6a) into (3.5b) and integrating gives

$$S_1(\xi,\eta,\tilde{t}) = G(\eta,\tilde{t})\,e^{\xi/2}+g_1(\xi,\tilde{t}), \tag{3.6b}$$

where $G(\eta, \tilde{t})$ is defined by

$$\partial G/\partial \eta = \tfrac{1}{2} f_1(\eta, \tilde{t}). \tag{3.7}$$

The solution to $O(\epsilon)$ involves the two arbitrary functions f_1 and g_1, and these must be determined next by requiring the $O(\epsilon^2)$-terms in (3.3) to be consistent.

We again introduce the characteristic dependent variables

$$R_2 = h_2 - u_2; \quad S_2 = h_2 + u_2, \tag{3.8a, b}$$

and rewrite (3.3*a*, *b*) as

$$R_{2_\xi} - \tfrac{1}{2}R_2 = \tfrac{1}{2}R_{1_{\tilde{t}}} + \tfrac{1}{8}(4\alpha + S_1 - 3R_1)(R_{1_\eta} + R_{1_\xi}) + \tfrac{1}{8}(S_1 + R_1)(S_{1_\eta} + S_{1_\xi}) + \tfrac{1}{8}\alpha(S_1 - R_1) - \tfrac{1}{2}[\tfrac{1}{4}(S_1 + 3R_1)]^2, \tag{3.9a}$$

$$S_{2_\eta} - \tfrac{1}{2}R_2 = -\tfrac{1}{2}S_{1_{\tilde{t}}} - \tfrac{1}{8}(4\alpha + 3S_1 - R_1)(S_{1_\eta} + S_{1_\xi}) + \tfrac{1}{8}(S_1 + R_1)(R_{1_\eta} + R_{1_\xi}) + \tfrac{1}{8}\alpha(S_1 - R_1) - \tfrac{1}{2}[\tfrac{1}{4}(S_1 + 3R_1)]^2. \tag{3.9b}$$

In order to obtain a consistency condition for the solution of (3.9*a*) we multiply both sides of this equation by the integrating factor $e^{-\xi/2}$. We then substitute (3.6*a*, *b*) into the resulting equation and (3.9*b*) to obtain

$$\frac{\partial(R_2 e^{-\xi/2})}{\partial \xi} = \tfrac{1}{2}(f_{1_{\tilde{t}}} + \alpha f_{1_\eta} + \tfrac{1}{4}\alpha f_1 + \tfrac{1}{4}\alpha G) + \mathrm{DST} + \mathrm{NST1}, \tag{3.10a}$$

$$\frac{\partial S_2}{\partial \eta} = \tfrac{1}{2}R_2 - \tfrac{1}{2}[g_{1_{\tilde{t}}} + (\alpha + \tfrac{3}{4}g_1)g_{1_\xi} + \tfrac{1}{16}g_1^2 - \tfrac{1}{4}\alpha g_1] + \mathrm{NST2}. \tag{3.10b}$$

Here NST1 only involves terms such as $A_1(\xi, \tilde{t})B_1(\eta, \tilde{t})$ where $A_1(\xi, \tilde{t})$ has a zero average over ξ, whereas NST2 represents terms of the form $A_2(\xi, \tilde{t})B_2(\eta, \tilde{t})$ with $B_2(\eta, \tilde{t})$ having a zero average over η. Also, $\mathrm{DST} = \tfrac{1}{8}e^{-\xi/2}(g_1 g_{1_\xi} + \alpha g_1 - \tfrac{1}{4}g_1^2)$ and this term is a function of ξ and $\tilde{t}$. As argued in Yu (1988) the terms in parentheses on right-hand side of (3.10*a*) will contribute terms proportional to ξ in the solution for R_2 upon integration. Such terms are inconsistent with a uniformly valid expansion over $O(\epsilon^{-1})$ intervals in x and t, and must therefore be eliminated by requiring

$$f_{1_{\tilde{t}}} + \alpha f_{1_\eta} + \tfrac{1}{4}\alpha f_1 + \tfrac{1}{4}\alpha G = 0. \tag{3.11}$$

Terms in DST will not contribute any inconsistent terms to R_2 after (3.10*a*) is integrated. However, we do keep track of these terms because they contribute functions which depend on ξ and $\tilde{t}$ but not η in the term $\tfrac{1}{2}R_2$ that appears on the right-hand side of (3.10*b*). To avoid any inconsistency in the solution for S_2 after (3.10*b*) is integrated, we must eliminate functions of the ξ and $\tilde{t}$ variables only in R_2 together with the bracketed terms on right-hand side of (3.10*b*). This requires

$$g_{1_{\tilde{t}}} + (\alpha + \tfrac{3}{4}g_1)g_{1_\xi} - \tfrac{1}{4}\alpha g_1 - \tfrac{1}{8}\alpha\, e^{\xi/2}\left(\int^{\xi} g_1 e^{-s/2}\, ds + C_1\right) = 0, \tag{3.12}$$

where C_1 is an integration constant.

If the initial conditions $u_0(x)$ and $h_0(x)$ in (1.2) are $2l$-periodic functions of x then g_1 will be periodic in ξ, and it follows from (3.6*a*, *b*) that if we define

$$f_1^*(\eta, \tilde{t}) = f_1(\eta, \tilde{t})\, e^{\eta/2}; \quad G^*(\eta, \tilde{t}) = G(\eta, \tilde{t})\, e^{\eta/2}, \tag{3.13a, b}$$

then f_1^* is periodic in η. Equation (3.11) now becomes

$$f_{1_{\tilde{t}}}^* + \alpha f_{1_\eta}^* - \tfrac{1}{4}\alpha f_1^* + \tfrac{1}{8}\alpha\, e^{\eta/2}\left(\int^{\eta} f_1^* e^{-s/2}\, ds + C_2\right) = 0, \tag{3.14}$$

where C_2 is a constant of integration and we have replaced $G(\eta, \tilde{t})$ by (3.7). We use the periodicity conditions to determine C_1, C_2 and the lower limit of the integral term in (3.12) and (3.14) to obtain

$$f^*_{1_{\tilde{t}}} + \alpha f^*_{1_\eta} = \tfrac{1}{4}\alpha f^*_1 - \tfrac{1}{8}\alpha\, e^{\eta/2}\left(\int_{-l}^{\eta} f^*_1\, e^{-s/2}\, ds + \frac{e^l}{1-e^l}\int_{-l}^{l} f^*_1\, e^{-s/2}\, ds\right), \tag{3.15a}$$

$$g_{1_{\tilde{t}}} + (\alpha + \tfrac{3}{4}g_1)\, g_{1_\xi} = \tfrac{1}{4}\alpha g_1 + \tfrac{1}{8}\alpha\, e^{\xi/2}\left(\int_{-l}^{\xi} g_1\, e^{-s/2}\, ds + \frac{e^l}{1-e^l}\int_{-l}^{l} g_1\, e^{-s/2}\, ds\right). \tag{3.15b}$$

These are the evolution equations for f^*_1 and g_1. Since the contribution from f^*_1 to h and u must be multiplied by $e^{(\xi-\eta)/2} = e^{-t}$, which decays exponentially in time, only the evolution equation (3.15*b*) is important for large t. Equation (3.15*b*) is a quasi-linear integro–partial differential equation in ξ and $\tilde{t}$ for g_1. Equivalently, multiplying (3.15*b*) by $e^{-\xi/2}$ and taking the partial derivative of the resulting equation with respect to ξ gives a second-order partial differential equation for g_1. Through (3.1), (3.4), (3.6) and (3.13), we find that initial conditions (1.2*a*, *b*) imply that

$$f^*_1(x, 0) = h_0(x) - u_0(x), \tag{3.16a}$$

$$g_1(x, 0) = h_0(x) + u_0(x) - G^*(x, 0), \tag{3.16b}$$

where $G^*(\eta, \tilde{t})$ is defined in (3.13*b*) and can be determined by (3.15*a*). In fact, by comparing (3.11) with (3.15*a*) we see that

$$G^*(\eta, \tilde{t}) = \tfrac{1}{2} e^{\eta/2}\left(\int_{-l}^{\eta} f^*_1\, e^{-s/2}\, ds + \frac{e^l}{1-e^l}\int_{-l}^{l} f^*_1\, e^{-s/2}\, ds\right). \tag{3.17}$$

Equations (3.15*a*, *b*) subject to the initial conditions (3.16*a*, *b*) determine f^*_1 and g_1 uniquely. Once the expressions for f^*_1 and g_1 are derived, the solution for h and u to order ϵ is then available through (3.13), (3.6), (3.4) and (3.1) as

$$h(x, t; \epsilon) = 1 + \tfrac{1}{2}\epsilon(g_1(\xi, \tilde{t}) + [G^*(\eta, \tilde{t}) + f^*_1(\eta, \tilde{t})]\, e^{(\xi-\eta)/2}) + O(\epsilon^2), \tag{3.18a}$$

$$u(x, t; \epsilon) = 2 + \epsilon\{\alpha + \tfrac{1}{2}[g_1(\xi, \tilde{t}) + [G^*(\eta, \tilde{t}) - f^*_1(\eta, \tilde{t})]\, e^{(\xi-\eta)/2}]\} + O(\epsilon^2). \tag{3.18b}$$

In order to interpret this result let us consider first the terms multiplied by $e^{(\xi-\eta)/2}$ in the expressions for h_1 and u_1. Each of these terms has the form of a periodic function of η, which is also slowly varying (a function of $\tilde{t}$), multiplying $e^{(\xi-\eta)/2} = e^{-t}$. Therefore, these terms decay exponentially as $t \to \infty$ along a ray $\eta = \text{const}$. In fact, these terms become unimportant when $t = O(\log(\epsilon^{-1}))$ which is small compared to the time $t = O(\epsilon^{-1})$ over which the expansion (3.18) is valid. Along a ray $\xi = \text{const.}$, i.e. if $x - 3t = \text{const.}$ as $t \to \infty$, the $\tfrac{1}{2}g_1(\xi, \tilde{t})$ term appearing in u_1 and h_1 represents a slowly varying contribution which remains of $O(1)$ and obeys the evolution equation (3.15*b*). As we shall see for specific examples, solutions of this evolution equation exhibit the typical long-term effects of weak nonlinearities including the steepening of waves and the eventual formation of bores. In particular, the amplitude of this wave grows with time at the initial rate predicted by the linear theory. This growth persists until the amplitude approximately doubles in magnitude, at which point the nonlinear terms come into play keeping the magnitude of the amplitude approximately constant even for $t = O(\epsilon^{-1})$.

An important feature of the solution is that for periodic initial values $g_1(\xi, 0)$ with a given wavelength $2l$, the solution of (3.15*b*), $g_1(\xi, \tilde{t})$, evolves with $\tilde{t}$ but remains periodic in ξ with the same $2l$ wavelength for all $\tilde{t} > 0$. To see this let us first consider the solution of (3.15*b*) at a neighbouring time $\tilde{t} = \Delta \ll 1$ by the method of characteristics for given $2l$-periodic initial data $g_1(\xi, 0)$. The periodicity condition $g_1(\xi_0+2l, 0) = g_1(\xi_0, 0)$ immediately implies that the characteristics emerging from ξ_0+2l and ξ_0 have the same slope $\mathrm{d}\xi/\mathrm{d}\tilde{t} = \alpha+\frac{3}{4}g_1(\xi_0, 0)$. It is easily verified that the periodicity condition also implies that the right-hand side of (3.15*b*) at $\tilde{t} = 0$ has the same value at the two points ξ_0+2l and ξ_0. Thus, identical characteristic equations and initial values govern the evolution of g_1 along the parallel characteristics emerging from ξ_0+2l and ξ_0. It then follows that the $2l$-periodicity of $g_1(\xi, \Delta)$ is preserved, and repeating this construction proves that $g_1(\xi, \tilde{t})$ remains $2l$-periodic for all $\tilde{t} > 0$. One can apply the basic idea of this proof for the exact problem (1.1) to argue that the initial wavelength is exactly conserved for finite-amplitude periodic disturbances.

To illustrate the above ideas, we again assume that the initial conditions (1.2*a, b*) are in the simple form $(l = \pi)$

$$h(x, 0; \epsilon) = 1-\epsilon \sin x; \quad u(x, 0; \epsilon) = F, \tag{3.19a, b}$$

so that $h_0(x) = -\sin x$ and $u_0(x) = 0$, and (3.16*a, b*) become

$$f_1(x, 0) = -\mathrm{e}^{-x/2} \sin x; \quad g_1(x, 0) = -\sin x - G(x, 0)\,\mathrm{e}^{x/2}. \tag{3.20a, b}$$

In this case, (3.11) has a solution in the form

$$f_1(\eta, \tilde{t}) = -\mathrm{e}^{a\alpha\tilde{t}-\eta/2} \sin(\eta+b\alpha\tilde{t}), \tag{3.21}$$

where a and b are undetermined coefficients. From (3.7) we have

$$G(\eta, \tilde{t}) = \mathrm{e}^{a\alpha\tilde{t}-\eta/2}\,\tfrac{1}{5}(\sin(\eta+b\alpha\tilde{t})+2\cos(\eta+b\alpha\tilde{t})). \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.11), collecting the coefficients in front of $\mathrm{e}^{a\alpha\tilde{t}-\eta/2}\cos(\eta+b\alpha\tilde{t})$ and $\mathrm{e}^{a\alpha\tilde{t}-\eta/2}\sin(\eta+b\alpha\tilde{t})$ we find that $a = \frac{3}{10}$ and $b = -\frac{9}{10}$. Therefore (3.21) and (3.22) become

$$f_1(\eta, \tilde{t}) = -\mathrm{e}^{3\alpha\tilde{t}/10-\eta/2} \sin(\eta-\tfrac{9}{10}\alpha\tilde{t}), \tag{3.23}$$

and

$$G(\eta, \tilde{t}) = \mathrm{e}^{3\alpha\tilde{t}/10-\eta/2}\,\tfrac{1}{5}(\sin(\eta-\tfrac{9}{10}\alpha\tilde{t})+2\cos(\eta-\tfrac{9}{10}\alpha\tilde{t})). \tag{3.24}$$

When (3.24) is used, the initial condition (3.20*b*) becomes

$$g_1(x, 0) = -\tfrac{1}{5}(6\sin x+2\cos x). \tag{3.25}$$

We now solve equation (3.15*b*) subject to the initial condition (3.25) by an explicit finite-difference method. A quadrature with repeated trapezoidal rule is used to evaluate the integral terms in (3.15*b*). The jump conditions across a discontinuity of the solution of (3.15*b*) can be obtained from the exact bore conditions for (1.1). Proceeding as in Yu (1988), we find that

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tilde{t}} = \alpha+\tfrac{3}{8}(g_1^+ + g_1^-), \tag{3.26}$$

where the plus and minus superscripts indicate values on either side of the discontinuity. Thus, the correct divergence form for (3.15*b*) is

$$(\alpha+\tfrac{3}{4}g_1)_{\tilde{t}}+[\tfrac{1}{2}(\alpha+\tfrac{3}{4}g_1)^2]_\xi = \frac{3\alpha}{16}g_1+\frac{3\alpha}{32}\mathrm{e}^{\xi/2}\left(\int_{-\pi}^{\xi} g_1\,\mathrm{e}^{-s/2}\,\mathrm{d}s+\frac{\mathrm{e}^{\pi}}{1-\mathrm{e}^{\pi}}\int_{-\pi}^{\pi} g_1\,\mathrm{e}^{-s/2}\,\mathrm{d}s\right). \tag{3.27}$$

In order to verify the accuracy of the asymptotic results, we numerically integrate the exact problem (1.1). As discussed in Yu (1988), we rescale the characteristic variables and solve the resulting equations also by an explicit finite-difference

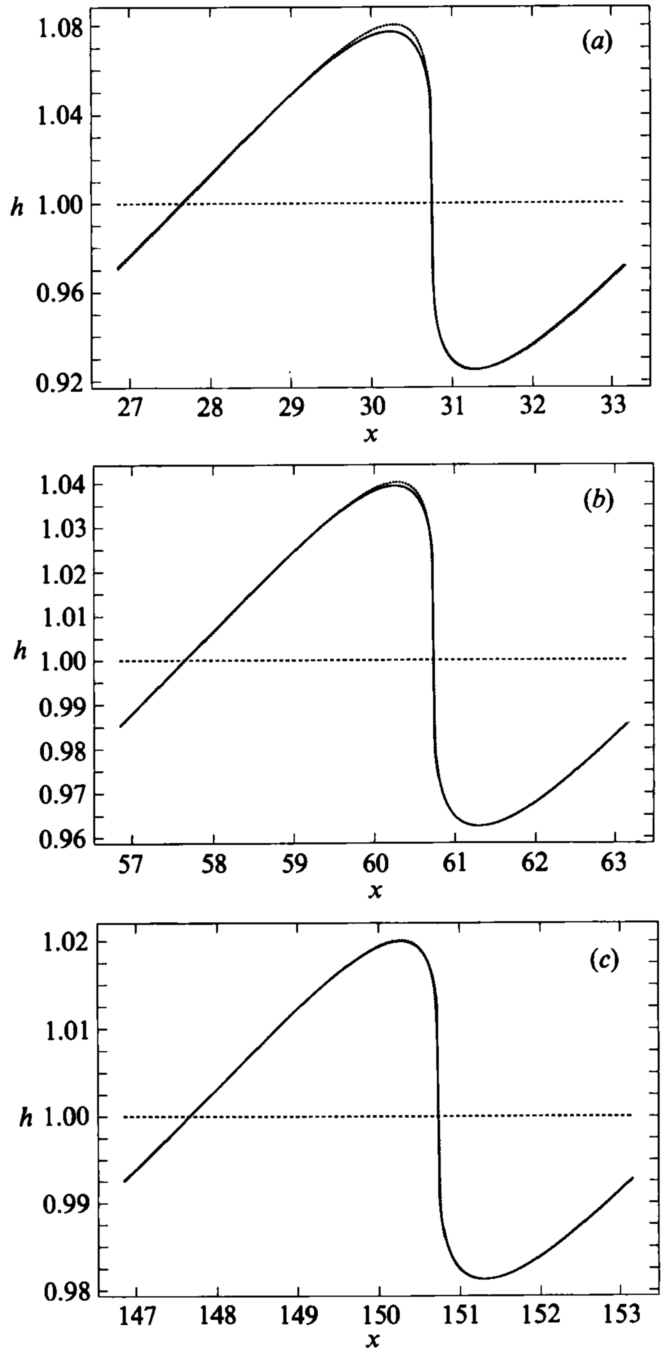

FIGURE 2. Numerical (——) and asymptotic (······) solutions for the periodic initial conditions $h(x,0;\epsilon) = 1-\epsilon\sin x; u(x,0;\epsilon) = F$: (*a*) $F = 2.1$, ($\epsilon = 0.1$, $\alpha = 1$) at $t = 10 = 1/\epsilon$; (*b*) $F = 2.05$, ($\epsilon = 0.05$, $\alpha = 1$) at $t = 20 = 1/\epsilon$; (*c*) $F = 2.02$, ($\epsilon = 0.02$, $\alpha = 1$) at $t = 50 = 1/\epsilon$.

method. In each of the algorithms used for solving (1.1) and (3.15*b*) we rely on the so called 'flux-splitting' technique and the time steps are chosen to satisfy the CFL condition to ensure stability. In figure 2(*a*) we show the theoretical (dotted curve) and numerical (solid curve) values of h in a 2π-interval in x for $t = 10$, $\epsilon = 0.1$ and

$F = 2.1$. A bore is about to form near $x = 30.75$. The maximum error everywhere away from this location is 0.5×10^{-2} and is certainly of $O(\epsilon^2)$. Near the incipient bore location the maximum error is about 1.8×10^{-2} which is between $O(\epsilon)$ and $O(\epsilon^2)$ and is entirely due to the error in the potential bore location. In order to ascertain that the theory is indeed accurate to $O(\epsilon)$ for times of $O(\epsilon^{-1})$ we investigate the accuracy of our results for two smaller values of ϵ. Figure 2(*b*) shows h for the case $F = 2.05$, $\epsilon = 0.05$ and $t = 20$. The maximum error away from the incipient bore location in this case is 2.4×10^{-3} and the maximum error near that area is 2.9×10^{-2}. A comparison of these errors with the numerical values $\epsilon = 5.0 \times 10^{-2}$ and $\epsilon^2 = 2.5 \times 10^{-3}$ shows that the accuracy obtained is consistent. Figure 2(*c*) is for the case where $F = 2.02$, $\epsilon = 0.02$ and $t = 50$. The two maximum errors are now 3.0×10^{-4} and 1.0×10^{-3}, respectively. These are also consistent with our theory and one observes a steady decrease of the error as we decrease the value of ϵ. A quasi-steady-state solution exists in all three cases and we will analyse this in the next section.

4. Quasi-steady-state solutions

In preparation for verifying our asymptotic solution in the limit $t \to \infty$, we summarize Dressler's (1949) results and append a discussion on how to connect his quasi-steady solutions to given initial conditions

Dressler proved that no continuous, periodic quasi-steady solutions exist for the system (1.1). However, for any given wavelength and progressing speed c, there exists a unique discontinuous periodic quasi-steady solution. Each cycle of the resulting roll wave pattern consists of two 'special' (cf. the discussion following (4.6)) continuous solutions joined by a bore as shown schematically in figure 3. Because Dressler was only interested in quasi-steady solutions, he did not address the question of how a given initial state evolves to the final quasi-steady state. For periodic initial conditions, for example, we have shown that the quasi-steady wavelength is exactly equal to the initial wavelength, but further detailed study is needed to relate c to the initial conditions. In particular, we derive next a constraint linking the average values of the initial and quasi-steady surface heights. This relation will uniquely define c once the form of the quasi-steady solution is known.

We modify Dressler's notation and use our dimensionless variables x and h such that the wavelength is $2l$ and the undisturbed height is 1. Since Dressler's solution is steady in a uniformly moving frame of reference, we make the change of variables $t^* = t$ and $\zeta = x - ct$, where c is the progressing speed. In this moving frame the velocity u^* and height h^* are given by $u^* = u - c$ and $h^* = h$. We use an overbar to denote the average of a variable over one wavelength of the quasi-steady solution, i.e.

$$\overline{[\,]} = \frac{1}{2l}\int_0^{2l} [\,]\,\mathrm{d}\zeta. \tag{4.1}$$

Averaging the law of mass conservation

$$h_{t^*} + (u^* h)_\zeta = 0, \tag{4.2}$$

and using the periodicity condition of u^*h, we find that $\bar{h}_{t^*} = 0$. When we use the initial condition (1.2*a*) we obtain the following physically obvious constraint for $\bar{h}(t^*)$:

$$\bar{h}(t^*) \equiv \text{const.} = 1 + \epsilon \bar{h}_0. \tag{4.3}$$

This is an exact (rather than asymptotic) condition valid for arbitrary ϵ. Note

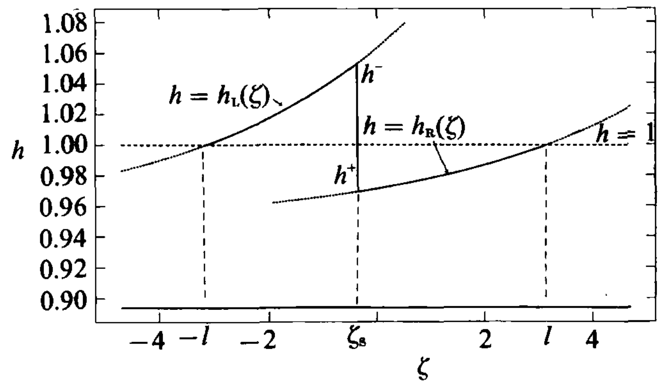

FIGURE 3. Roll wave.

incidentally that averaging the momentum conservation law does not provide an analogous constraint for $\bar{u}^*(t^*)$. We have

$$(u^*h)_{t^*}+(u^{*2}h+\tfrac{1}{2}h^2)_\zeta = h-(u^*+c)^2/F^2, \tag{4.4}$$

and using the periodicity condition of $u^{*2}h+\frac{1}{2}h^2$, we find that

$$\overline{(u^*h)_{t^*}} = 1+\epsilon\bar{h}_0-\overline{(u^*+c)^2}/F^2. \tag{4.5}$$

Since the right-hand side of (4.5) is, in general, not zero, the average momentum is, in general, a function of time and this is due to the friction and gravitational forces.

We next discuss how to use (4.3) to isolate the particular quasi-steady solution that is selected for a given initial wave. It is shown in Dressler (1949) that in each cycle of a roll wave there exists a point ζ_c where the flow is critical, i.e.

$$u_c^{*2} = h_c, \tag{4.6a}$$

and that the following condition is necessary for a consistent solution:

$$(u_c^*+c)^2 = F^2h_c. \tag{4.6b}$$

Dressler denotes flows which satisfy the two conditions (4.6*a*, *b*) as 'special' solutions. The two equations (4.6*a*, *b*) involve the three unknowns u_c^*, h_c and c. We shall use the constraint (4.3) to close the system. Since (4.3) involves the special solution $h(\zeta)$, we need to first define this function in detail.

The special solution defining the left continuous segment $h_L(\zeta)$ is given by the inverse of

$$x-ct \equiv \zeta = Q(h) = h-1+k_1\log\left(\frac{h-h_A}{1-h_A}\right)-k_2\log\left(\frac{h-h_B}{1-h_B}\right)-l. \tag{4.7}$$

Here, the constants h_A, h_B are defined by

$$h_A = \frac{2F+1+(4F+1)^{\frac{1}{2}}}{2F^2}h_c;\quad h_B = \frac{2F+1-(4F+1)^{\frac{1}{2}}}{2F^2}h_c, \tag{4.8a, b}$$

and satisfy the inequalities $0 < h_B < h_A < h_c$ for $F > 2$. The constants k_1 and k_2 are given by

$$k_1 = \frac{h_A^2+h_Ah_c+h_c^2}{h_A-h_B};\quad k_2 = \frac{h_B^2+h_Bh_c+h_c^2}{h_A-h_B}. \tag{4.9a, b}$$

We also note that the left end of the wave is chosen at $\zeta = -l$; thus $Q(1) = -l$ in (4.7), i.e. $h_L(-l) = 1$. The right continuous segment of the wave, $h_R(\zeta)$, is defined by the inverse of $\zeta = 2l+Q(h)$, hence it satisfies the periodicity condition $h_R(l) = 1$.

The two jump conditions across a bore are given by

$$u^{*-}h^- = u^{*+}h^+ = u_c^* h_c, \tag{4.10a}$$

$$h^+ = \frac{1}{2}\left\{\left[(h^-)^2 + \frac{8h_c^3}{h^-}\right]^{\frac{1}{2}} - h^-\right\}, \tag{4.10b}$$

where the plus and minus superscripts indicate values on the right and left sides respectively of the bore. The first condition (4.10*a*) states that the progressive discharge rate is constant. In deriving the solution (4.7) we have used this condition to eliminate the dependent variable u^*. Therefore, (4.10*a*) is always satisfied. We use the second condition (4.10*b*) to locate the bore as follows. At the bore location $\zeta = \zeta_s$, we have $\zeta_s = Q(h^-) = 2l + Q(h^+)$, i.e.

$$k_1 \log\frac{h^- - h_A}{h^+ - h_A} - k_2 \log\frac{h^- - h_B}{h^+ - h_B} - 2l + h^- - h^+ = 0. \tag{4.11}$$

The values of h^- and h^+ are defined by (4.10*b*) and (4.11) implicitly. Once these values are found, we use $\zeta_s = Q(h^-)$ or $\zeta_s = 2l + Q(h^+)$ to calculate the bore location ζ_s and this defines $h(\zeta)$ completely (see figure 3).

We are now ready to evaluate the three unknown constants u_c^*, h_c and c that are associated with a given initial wave. From (4.3) we have

$$\frac{1}{2l}\int_{\zeta_s - 2l}^{\zeta_s} h_L(\zeta)\,\mathrm{d}\zeta = 1 + \epsilon\bar{h}_0. \tag{4.12}$$

Integrating by parts and substituting $\zeta = Q(h)$ into the result gives

$$h^+ + \frac{1}{2l}(h^- - h^+)\,\zeta_s - \frac{1}{2l}\int_{h^+}^{h^-} Q(h)\,\mathrm{d}h = 1 + \epsilon\bar{h}_0, \tag{4.13}$$

where $Q(h)$ is defined by (4.7). For a given initial wave $\bar{h}_0$, (4.13) involves the unknown constant h_c only and can be used to calculate h_c uniquely. A numerical value for h_c is easily computed; however, an explicit analytic expression is not possible. Once h_c is available, u_c^* follows from (4.6*a*) and c from (4.6*b*). This completely defines one cycle of the roll wave which is the periodic extension of this cycle. This result is exact (valid for arbitrary ϵ) but not analytic. It is possible to compute an analytic expression for the roll wave and all the parameters that define it if we restrict ϵ to be small and use our asymptotic results as discussed below.

We begin with a derivation of the quasi-steady limit of the asymptotic solution correct to $O(\epsilon)$ given by (3.18*a*, *b*). As was pointed out earlier, the amplitude of the disturbance along the η = const. ray decays exponentially. Therefore, as $t \to \infty$ these terms vanish and (3.18*a*, *b*) become

$$h = 1 + \tfrac{1}{2}\epsilon g^*; \quad u = 2 + \epsilon(\alpha + \tfrac{1}{2}g^*), \tag{4.14a, b}$$

where g^* is the quasi-steady value of g_1, and must be obtained from (3.15*b*) and (3.26). In order to compare with Dressler's solution, we expand the progressing speed c as $c = 3 + \epsilon\beta + O(\epsilon^2)$ and make the change of variables $\tilde{t}^* = \tilde{t}$ and $\zeta = \xi - \beta\tilde{t}$ in (3.15*b*) and (3.26). At this stage β is an unknown $O(1)$ constant. It can be shown that the integral terms in (3.15*b*) are unchanged under this transformation. By letting $\tilde{t}^* \to \infty$ while holding ζ fixed, one obtains the following equation for $g^*(\zeta)$:

$$(\alpha - \beta + \tfrac{3}{4}g^*)\frac{\mathrm{d}g^*}{\mathrm{d}\zeta} = \tfrac{1}{4}\alpha g^* + \tfrac{1}{8}\alpha\,\mathrm{e}^{\zeta/2}\left(\int_{-l}^{\zeta} g^*\,\mathrm{e}^{-s/2}\,\mathrm{d}s + \frac{\mathrm{e}^l}{1 - \mathrm{e}^l}\int_{-l}^{l} g^*\,\mathrm{e}^{-s/2}\,\mathrm{d}s\right), \tag{4.15}$$

and the following jump condition:

$$\alpha-\beta+\tfrac{3}{8}(g^{*+}+g^{*-}) = 0. \tag{4.16}$$

Taking the derivative of (4.15) with respect to ζ and eliminating the integral terms between the resulting equation and (4.15) shows that g^* must satisfy

$$\frac{\mathrm{d}^2(16(\alpha-\beta)g^*+6g^{*2})}{\mathrm{d}\zeta^2} = \frac{\mathrm{d}(3g^{*2}+4(3\alpha-2\beta)g^*)}{\mathrm{d}\zeta}. \tag{4.17}$$

Integrating once, we have

$$(\alpha-\beta+\tfrac{3}{4}g^*)\frac{\mathrm{d}g^*}{\mathrm{d}\zeta} = \tfrac{3}{16}[g^{*2}+\tfrac{4}{3}(3\alpha-2\beta)g^*+K_1], \tag{4.18}$$

where K_1 is a constant of integration and can be determined from the condition (4.6*a*). We use (4.14*a, b*) to expand (4.6*a*) and obtain

$$g_c^* = \tfrac{4}{3}(\beta-\alpha), \tag{4.19}$$

where g_c^* is the value of g^* at which the flow is critical. At the critical point the left-hand side of (4.18) vanishes. A consistency argument similar to that given by Dressler (1949) shows that the necessary condition for existence of the special solution is that the right-hand side also vanishes. This implies that

$$K_1 = \tfrac{16}{9}(\beta-\alpha)(\beta-2\alpha), \tag{4.20}$$

and (4.18) reduces to

$$\frac{\mathrm{d}g^*}{\mathrm{d}\zeta} = \tfrac{1}{4}[g^*-\tfrac{4}{3}(\beta-2\alpha)]. \tag{4.21}$$

The solution of this equation subject to $g^*(-l) = 0$ is given by

$$\zeta = -l+4\log\left|1+\frac{3g^*}{4(2\alpha-\beta)}\right|, \tag{4.22}$$

and the right segment of the wave can be obtained from (4.22) with a $2l$-shift, i.e.

$$\zeta = l+4\log\left|1+\frac{3g^*}{4(2\alpha-\beta)}\right|. \tag{4.23}$$

At a bore location ζ_s (4.22) and (4.23) imply

$$-l+4\log\left|1+\frac{3g^{*-}}{4(2\alpha-\beta)}\right| = l+4\log\left|1+\frac{3g^{*+}}{4(2\alpha-\beta)}\right|. \tag{4.24}$$

Solving this equation together with (4.16) gives

$$g^{*+} = \tfrac{4}{3}[\beta-2\alpha/(1+\mathrm{e}^{-l/2})]; \quad g^{*-} = \tfrac{4}{3}[\beta-2\alpha/(1+\mathrm{e}^{l/2})]. \tag{4.25a, b}$$

Substituting (4.25*a*) into (4.23) we obtain the bore location as

$$\zeta_s = -4\log[(2\alpha-\beta)/\alpha]\cosh\tfrac{1}{4}l]. \tag{4.25c}$$

The solution defined by (4.22)–(4.25) contains the unknown constant β that we introduced in the expansion of c. We determine β through the expansion of the additional constraint (4.3) using (4.14*a*), i.e.

$$\bar{g}^* \equiv \frac{1}{2l}\int_{\zeta_s-2l}^{\zeta_s} g^*\,\mathrm{d}\zeta = 2\bar{h}_0. \tag{4.26}$$

Solving for g^* from (4.22) and substituting the result into (4.26), one finds that

$$\beta = \tfrac{3}{2}\bar{h}_0+2\alpha[1-(2/l)\tanh\tfrac{1}{4}l]. \tag{4.27}$$

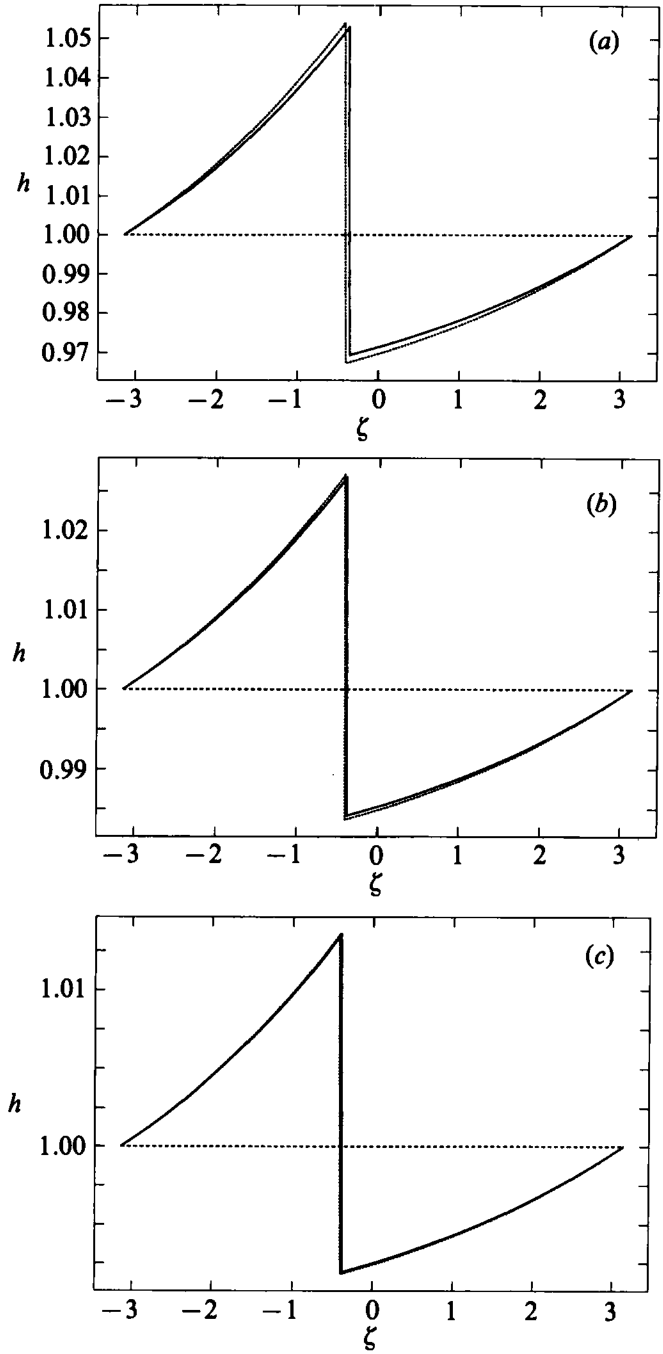

FIGURE 4. Numerical (——) and asymptotic (· · · · · ·) quasi-steady solutions for the periodic initial conditions with (*a*) $F = 2.1$, (*b*) $F = 2.05$, (*c*) $F = 2.02$.

The quasi-steady state of the asymptotic solution correct to $O(\epsilon)$ over a $2l$-interval is now available explicitly from (4.27), (4.22)–(4.25) and (4.14).

We compare the exact and asymptotic quasi-steady solutions derived above in figure 4 (*a–c*) for the three cases $F = 2.1$, 2.05 and 2.02, respectively. The solid curves represent the exact solution and the dotted curves are the asymptotic solution correct to $O(\epsilon)$. The values of these two solutions are shown over a 2π-interval in ζ and found to agree extremely well. Furthermore, the bore location is also very well predicted. The maximum errors for these three cases are 2.3×10^{-3}, 6.3×10^{-4} and 1.0×10^{-4}, respectively.

5. Arbitrary initial disturbance

Solutions corresponding to arbitrary initial conditions may be derived as the limiting forms of our results in §§3 and 4 for the case of infinite wavelength. In fact, with the exception of the evolution equations (3.15*a*, *b*), which are now in the form

$$f^*_{1_t}+\alpha f^*_{1_\eta} = \tfrac{1}{4}\alpha f^*_1-\tfrac{1}{8}\alpha\, e^{\eta/2}\int_\infty^\eta f^*_1\, e^{-s/2}\, ds, \qquad (5.1a)$$

$$g_{1_t}+(\alpha+\tfrac{3}{4}g_1)\, g_{1_\xi} = \tfrac{1}{4}\alpha g_1+\tfrac{1}{8}\alpha\, e^{\xi/2}\int_\infty^\xi g_1\, e^{-s/2}\, ds, \qquad (5.1b)$$

all the other formulae in §3 are unchanged. Actually, (5.1) is more general than (3.15) in the sense that if one assumes f_1^* and g_1 to be periodic functions of η and ξ respectively, one recovers (3.15). We verify the accuracy of the asymptotic results for the isolated initial disturbance given by

$$h(x,0;\epsilon) = 1-\epsilon\, e^{-\pi x^2}; \quad u(x,0;\epsilon) = F. \qquad (5.2a, b)$$

As for the results in figure 2, we compare our asymptotic solution with the numerical integration of the exact problem (1.1) in figure 5. In figure 5(*a*) we show the theoretical (dotted curve) and numerical (solid curve) values of h over an interval of about 10 in x for $t = 10$, $\epsilon = 0.1$ and $F = 2.1$. Near the incipient bore location the maximum error is about 4.0×10^{-2} which is between $O(\epsilon)$ and $O(\epsilon^2)$, and is entirely due to the error in the potential bore location. Everywhere away from this location, the maximum error is 0.3×10^{-2} and is certainly of $O(\epsilon^2)$. Figure 5(*b*) shows h for the case where $F = 2.05$, $\epsilon = 0.05$ and $t = 20$. The maximum error away from the incipient bore location in this case is 0.2×10^{-2} and the maximum error near that area is 2.0×10^{-2}. A comparison of these errors with the numerical values $\epsilon = 5.0\times10^{-2}$ and $\epsilon^2 = 0.25\times10^{-2}$ shows that the accuracy obtained is consistent. Figure 5(*c*) is for the case where $F = 2.02$, $\epsilon = 0.02$ and $t = 50$. The two maximum errors now 5.0×10^{-5} and 1.8×10^{-3}, respectively. These are also consistent with our theory and one observes a steady decrease of error as we decrease the value of ϵ.

Now, for the quasi-steady solutions corresponding to those in §4, it is convenient to locate the bore at the origin. In this coordinate system, Dressler's special solution defining the left continuous segment $h_L(\zeta)$ is given by the inverse of

$$x-ct \equiv \zeta = Q(h) = h-h^-+k_1\log\left(\frac{h-h_A}{h^--h_A}\right)-k_2\log\left(\frac{h-h_B}{h^--h_B}\right), \qquad (5.3)$$

where the constants h_A, h_B, k_1 and k_2 are still defined by (4.8) and (4.9). Equations (4.10*b*) and (4.11), which define the values of h^- and h^+ implicitly, are now in the form

$$h^+ = h_A; \quad h^- = \frac{1}{2}\left\{\left[(h^+)^2+\frac{8h_c^3}{h^+}\right]^{\frac{1}{2}}-h^+\right\}, \qquad (5.4a, b)$$

where the plus and minus superscripts indicate values on the right and left sides respectively of the bore. Equation (4.3) implies that we must integrate $h_L(\zeta)$ over an

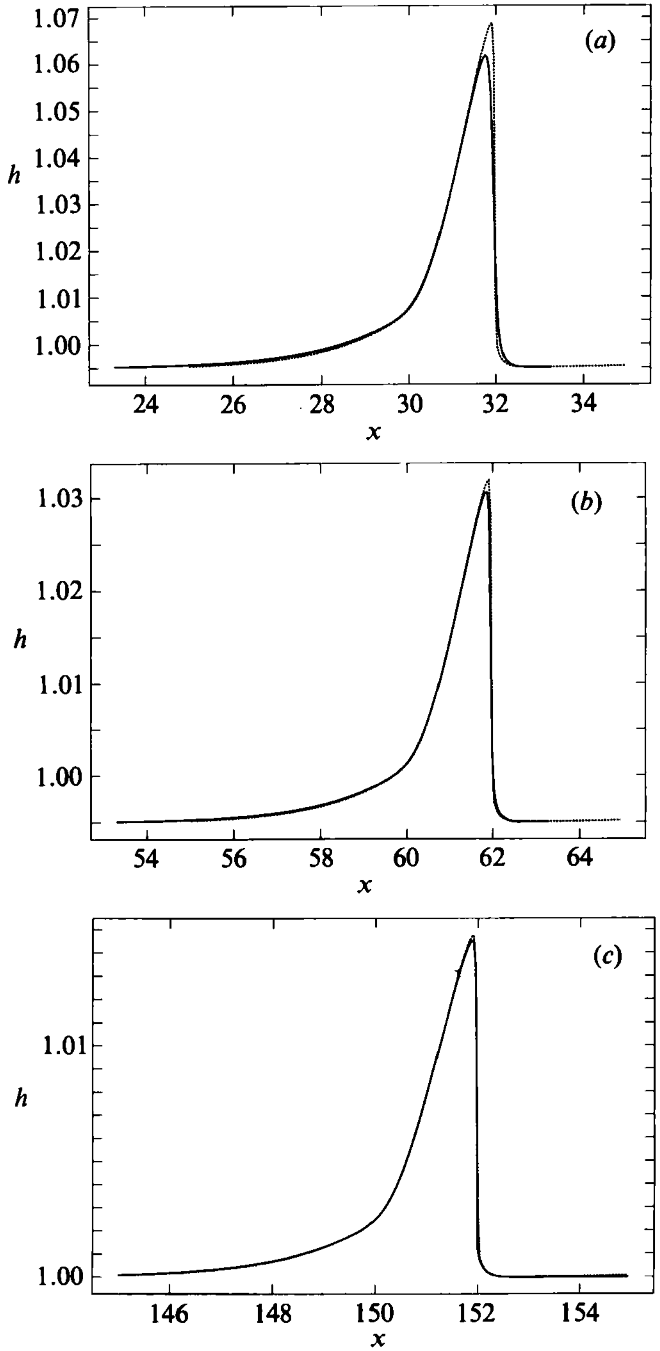

FIGURE 5. Numerical (——) and asymptotic (······) solutions for the isolated initial conditions $h(x,0;\epsilon) = 1-\epsilon\,e^{-\pi x^2}$; $u(x,0;\epsilon) = F$: (*a*) $F = 2.1$, ($\epsilon = 0.1$, $\alpha = 1$) at $t = 10 = 1/\epsilon$; (*b*) $F = 2.05$, ($\epsilon = 0.05$, $\alpha = 1$) at $t = 20 = 1/\epsilon$; (*c*) $F = 2.02$, ($\epsilon = 0.02$, $\alpha = 1$) at $t = 50 = 1/\epsilon$.

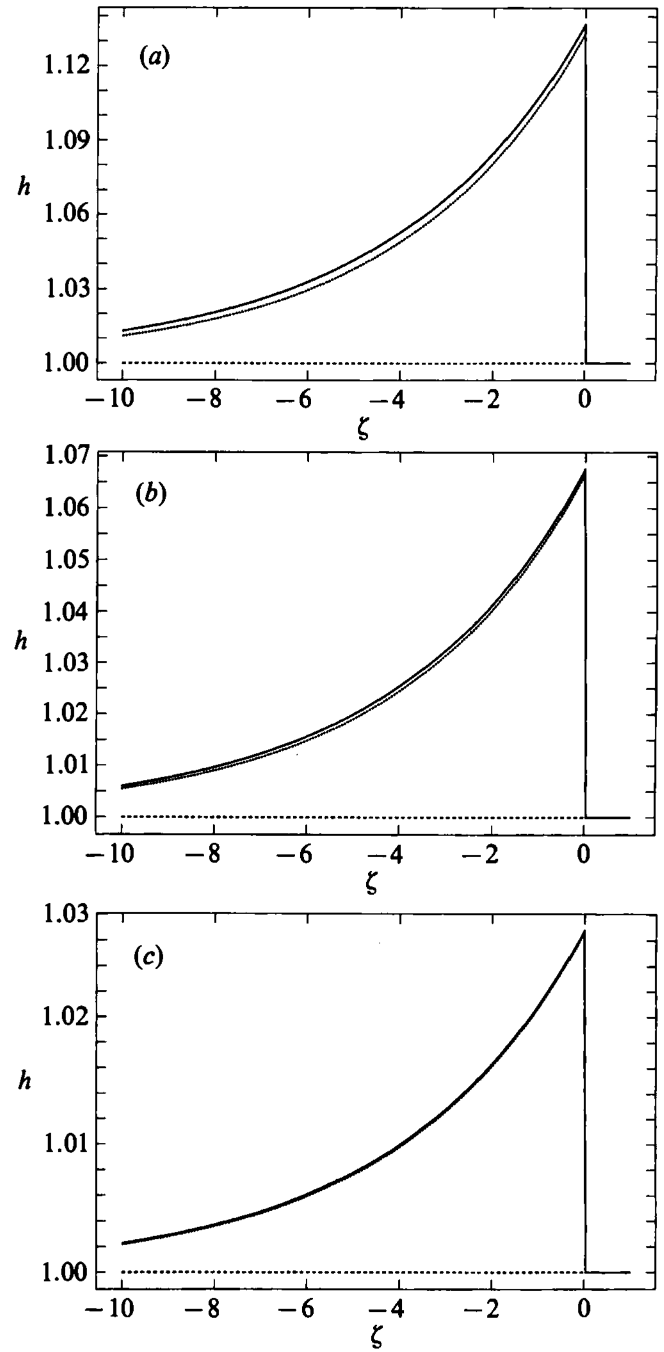

FIGURE 6. Numerical (——) and asymptotic (······) quasi-steady solutions for the isolated initial conditions with (a) $F = 2.1$, (b) $F = 2.05$, (c) $F = 2.02$.

interval of infinite length. We, therefore, must require $h \to 1$ as $\zeta \to -\infty$, that is $h^+ = h_A = 1$. From (4.8a) one finds that

$$h_c = \frac{2F^2}{2F+1+(4F+1)^{\frac{1}{2}}}. \tag{5.5}$$

Once these values are found, the solution to the left of the bore is available from (5.3). The solution to the right of the bore is the undisturbed water $h = 1$.

We derive next the quasi-steady limit of the asymptotic solution correct to $O(\epsilon)$ given by (3.13a, b) for the non-periodic case. From (4.21) we have

$$g^* = C\,e^{\zeta/4} + \tfrac{4}{3}(\beta - 2\alpha), \tag{5.6}$$

where C is the constant of integration. With the bore located at the origin, the additional constraint (4.26) now becomes

$$\lim_{l \to \infty} \frac{1}{2l} \int_{-2l}^{0} g^* \, d\zeta = 0. \tag{5.7}$$

This requires that the constant term in (5.6) vanishes, i.e. $\beta = 2\alpha$ and

$$g^* = C\,e^{\zeta/4}. \tag{5.8}$$

To determine the constant of integration C, we consider the jump condition (4.16). Since we have $g^* \to 0$ as $\zeta \to -\infty$, then $g^{*+} = 0$, and from (4.16) $g^{*-} = \frac{8}{3}\alpha$. Evaluating g^* at the origin where $\zeta = 0$ from (5.8) one finds that $C = \frac{8}{3}\alpha$. Therefore, to the left of the bore

$$g^* = \tfrac{8}{3}\alpha\,e^{\zeta/4}, \tag{5.9}$$

and to the right of the bore, we have undisturbed water, i.e. $g^* = 0$.

We compare the exact and asymptotic quasi-steady-state solutions derived above in figure 6(a–c) for the three cases $F = 2.1$, 2.05 and 2.02, respectively. The solid curves represent the exact solution and the dotted curves are the asymptotic solution correct to $O(\epsilon)$. The values of these two solutions are shown over an interval in ζ of about 11 in length and found to agree extremely well. The maximum errors for these three cases are 4.0×10^{-3}, 1.0×10^{-3} and 1.7×10^{-4}, respectively.

6. Concluding remarks

We have derived an asymptotic solution of the model equations (1.1) for arbitrary initial conditions (1.2) with ϵ small. This solution to $O(\epsilon)$ has the form (3.18) in which the functions f_1^* and g_1 obey the decoupled integro–partial differential evolution equations (3.15) for periodic initial disturbances (or the more general system (5.1) for arbitrary initial conditions).

We have verified that our results remain accurate, as required by a multiple scale analysis, over the time interval $[0, T(\epsilon)]$, where $T = O(\epsilon^{-1})$. It turns out that for $t < T$ the solution is time-dependent; a time on the order of $10T$ is required before the solution tends to a quasi-steady state.

Dressler pointed out that the quasi-steady-state solution is uniquely determined by its wavelength and the progressing speed. We show in addition that a given arbitrary periodic initial disturbance tends to the roll wave having the same wavelength, and that the progressing speed of this roll wave is uniquely defined by the average value only of the initial wave, independently of its actual shape. In the limit $t \to \infty$ our asymptotic solution confirms these results, and provides explicit formulae for the quasi-steady solution.

The remarkable accuracy of our asymptotic results for $t \rightarrow \infty$ is in part due to the fact that we compare only the limiting forms of both the asymptotic and exact equations. Also, we consider one cycle only of the roll wave, and we impose identical periodic boundary conditions on both solutions. A more stringent comparison would involve the 'exact' and asymptotic solution computed for the same initial state over a time interval that is sufficiently long to ensure a well-developed quasi-steady state. Such a calculation is impractical for the weakly unstable problem as it requires integrations over times equal to about $10\epsilon^{-1}$. At any rate, our results indicate that as $t \rightarrow \infty$, the asymptotic solution will, at worst, have an $O(\epsilon)$ phase shift relative to the 'exact' roll wave pattern; the actual roll wave profile is predicted very accurately.

Finally, we point out that inclusion of a term hu_{xx} multiplied by a small parameter in (1.1*b*) will have a significant effect only in a thin layer centred at a bore, and will smooth this discontinuity exactly as does the second derivative term in Burgers' equation (e.g. see Whitham 1974, Ch. 4 or Kevorkian 1990, §5.3.6). Since such a term is small, our perturbation analysis still holds, and we encounter second-derivative terms in the evolution equations (3.15). The numerical integration of these more general equations is not significantly altered.

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